

# A NOTE ON LAGRANGIAN COBORDISMS BETWEEN LEGENDRIAN SUBMANIFOLDS OF $\mathbb{R}^{2n+1}$

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**ABSTRACT.** We study the relation of an embedded Lagrangian cobordism between two closed, orientable Legendrian submanifolds of  $\mathbb{R}^{2n+1}$ . More precisely, we investigate the behavior of the Thurston-Bennequin number and (linearized) Legendrian contact homology under this relation. The result about the Thurston-Bennequin number can be considered as a generalization of the result of Chantraine which holds when  $n = 1$ . In addition, we provide a few constructions of Lagrangian cobordisms and prove that there are infinitely many pairs of Lagrangian cobordant and not Legendrian isotopic Legendrian  $n$ -tori in  $\mathbb{R}^{2n+1}$ .

## 1. INTRODUCTION

A *contact manifold*  $(M, \xi)$  is a  $(2n + 1)$ -dimensional manifold  $M$  equipped with a smooth maximally nonintegrable hyperplane field  $\xi \subset TM$ , i.e., locally  $\xi = \ker \alpha$ , where  $\alpha$  is a 1-form which satisfies  $\alpha \wedge (d\alpha)^n \neq 0$ .  $\xi$  is a *contact structure* and  $\alpha$  is a *contact 1-form* which locally defines  $\xi$ . The *Reeb vector field*  $R_\alpha$  of a contact form  $\alpha$  is uniquely defined by the conditions  $\alpha(R_\alpha) = 1$ ,  $d\alpha(R_\alpha, \cdot) = 0$ . The most basic contact manifold is  $(\mathbb{R}^{2n+1}, \xi)$ , where  $\mathbb{R}^{2n+1}$  has coordinates  $(x_1, y_1, \dots, x_n, y_n, z)$ , and  $\xi$  is given by  $\alpha = dz - \sum_{i=1}^n y_i dx_i$ . Note that  $R_\alpha = \partial z$ . From now on for ease of notation we write  $\mathbb{R}^{2n+1}$  instead of  $(\mathbb{R}^{2n+1}, \xi)$ .

A *Legendrian submanifold* of  $\mathbb{R}^{2n+1}$  is an  $n$ -dimensional submanifold  $\Lambda$  which is everywhere tangent to  $\xi$ , i.e.,  $T_x \Lambda \subset \xi_x$  for every  $x \in \Lambda$ . The *Lagrangian projection* is a map  $\Pi : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n}$  defined by

$$\Pi(x_1, y_1, \dots, x_n, y_n, z) = (x_1, y_1, \dots, x_n, y_n).$$

Moreover, for  $\Lambda$  in an open dense subset of all Legendrian submanifolds with  $C^\infty$  topology, the self-intersection of  $\Pi(\Lambda)$  consists of a finite number of transverse double points. Legendrian submanifolds which satisfy this property are called *chord generic*. A *Reeb chord* of  $\Lambda$  is a path along the flow of the Reeb vector field which begins and ends on  $\Lambda$ . Since  $R_\alpha = \partial z$ , there is a one-to-one correspondence between Reeb chords of  $\Lambda$  and double points of  $\Pi(\Lambda)$ . From now on we assume that all Legendrian submanifolds of  $\mathbb{R}^{2n+1}$  that we consider are connected and chord generic.

The *symplectization* of  $\mathbb{R}^{2n+1}$  is the symplectic manifold  $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha))$ , where  $t$  is a coordinate on  $\mathbb{R}$ .

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**Definition 1.1.** Let  $\Lambda_-$  and  $\Lambda_+$  be two Legendrian submanifolds of  $\mathbb{R}^{2n+1}$ . We say that  $\Lambda_-$  is Lagrangian cobordant to  $\Lambda_+$  if there exists a smooth cobordism  $(L; \Lambda_-, \Lambda_+)$ , and a Lagrangian embedding from  $L$  to  $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha))$  such that

$$\begin{aligned} L|_{(-\infty, -T_L] \times \mathbb{R}^{2n+1}} &= (-\infty, -T_L] \times \Lambda_-, \\ L|_{[T_L, \infty) \times \mathbb{R}^{2n+1}} &= [T_L, \infty) \times \Lambda_+ \end{aligned}$$

for some  $T_L \gg 0$  and  $L^c := L|_{[-T_L-1, T_L+1] \times \mathbb{R}^{2n+1}}$  is compact. In this case, we write  $\Lambda_- \prec_L^{lag} \Lambda_+$ . In the case of an exact Lagrangian embedding, we write  $\Lambda_- \prec_L^{ex} \Lambda_+$ . If there is an embedded cobordism  $L$  from  $\Lambda_-$  to  $\Lambda_+$ , we write  $\Lambda_- \prec_L \Lambda_+$ . We will in general not distinguish between  $L$  and  $L^c$  and call both  $L$ . If  $L_\Lambda$  is a filling of  $\Lambda$  in the symplectization of  $\mathbb{R}^{2n+1}$ , i.e.,  $L_\Lambda$  is an embedded cobordism with empty  $-\infty$ -boundary and  $+\infty$ -boundary  $\Lambda$ , then we write  $\emptyset \prec_{L_\Lambda} \Lambda$ . If  $L_\Lambda$  is given by Lagrangian embedding, then we write  $\emptyset \prec_{L_\Lambda}^{lag} \Lambda$ . In the case of an exact Lagrangian embedding, we write  $\emptyset \prec_{L_\Lambda}^{ex} \Lambda$ . From now on we assume that all Lagrangian cobordisms that we consider are connected.

For the discussion about Lagrangian cobordisms between Legendrian knots we refer to [3] and [9], and for the obstructions to the existence of Lagrangian cobordisms defined using the theory of generating families we refer to [16, 17].

We now say a few words about the Thurston-Bennequin invariant (number) in the case of Legendrian submanifolds of standard contact  $\mathbb{R}^{2n+1}$ .

Given a closed, orientable, connected Legendrian submanifold  $\Lambda$  of  $\mathbb{R}^{2n+1}$  we define an invariant, called the Thurston-Bennequin number of  $\Lambda$ . This invariant was originally defined by Bennequin [2] and independently by Thurston when  $n = 1$ , and was generalized to higher dimensions by Tabachnikov [19].

Pick an orientation on  $\Lambda \subset \mathbb{R}^{2n+1}$ . Push  $\Lambda$  slightly off of itself along  $R_\alpha = \partial z$  to get another oriented submanifold  $\Lambda'$  disjoint from  $\Lambda$ . The Thurston-Bennequin invariant of  $\Lambda$  is the linking number

$$tb(\Lambda) = lk(\Lambda, \Lambda').$$

Note that  $tb(\Lambda)$  is independent of the choice of orientation on  $\Lambda$  since changing it changes also the orientation of  $\Lambda'$ .

Our goal is to prove the following theorem:

**Theorem 1.2.** Let  $\Lambda_-$  and  $\Lambda_+$  be two closed, orientable Legendrian submanifolds of  $\mathbb{R}^{2n+1}$ .

(1) If  $n$  is even and  $\Lambda_- \prec_L \Lambda_+$ , then

$$tb(\Lambda_+) + tb(\Lambda_-) = (-1)^{\frac{n}{2}+1} \chi(L).$$

(2) If  $n$  is odd,  $\emptyset \prec_{L_\Lambda}^{ex} \Lambda_-$  and  $\Lambda_- \prec_L^{ex} \Lambda_+$ , then

$$tb(\Lambda_+) - tb(\Lambda_-) = (-1)^{\frac{(n-2)(n-1)}{2}+1} \chi(L).$$

*Remark 1.3.* Let  $\Lambda$  be a closed, orientable Legendrian submanifold of  $\mathbb{R}^{2n+1}$ .

(1) If  $n$  is even and  $\emptyset \prec_{L_\Lambda} \Lambda$ , then

$$tb(\Lambda) = (-1)^{\frac{n}{2}+1} \chi(L_\Lambda).$$

(2) If  $n$  is odd and  $\emptyset \prec_{L_\Lambda}^{ex} \Lambda$ , then

$$tb(\Lambda) = (-1)^{\frac{(n-2)(n-1)}{2}+1} \chi(L_\Lambda).$$

Legendrian contact homology was introduced by Eliashberg, Givental and Hofer in [12] and independently, for Legendrian knots in  $\mathbb{R}^3$ , by Chekanov [4]. We will briefly recall the definition of (linearized) Legendrian contact homology in Section 2. Following Ekholm [5], we observe that Lagrangian cobordism between two Legendrian submanifolds can be used to define a map between the Legendrian contact homology algebras, for details see [8].

In this paper, we establish the following two long exact sequences:

**Theorem 1.4.** *Let  $\Lambda_-$  and  $\Lambda_+$  be two closed, orientable Legendrian submanifolds of  $\mathbb{R}^{2n+1}$  such that  $\emptyset \prec_{L_{\Lambda_-}}^{ex} \Lambda_-$ . Then from the condition  $\Lambda_- \prec_L^{ex} \Lambda_+$  it follows that there is the following exact sequence*

$$\rightarrow H_i(\Lambda_-) \rightarrow H_i(L) \oplus LCH_{\varepsilon_-}^{n-i+2}(\Lambda_-) \rightarrow LCH_{\varepsilon_+}^{n-i+2}(\Lambda_+) \rightarrow H_{i-1}(\Lambda_-) \rightarrow .$$

*In addition,  $\Lambda_- \prec_L^{ex} \Lambda_+$  implies that there is the following exact sequence*

$$\rightarrow LCH_{\varepsilon_-}^{n-i+2}(\Lambda_-) \rightarrow LCH_{\varepsilon_+}^{n-i+2}(\Lambda_+) \rightarrow H_i(L, \Lambda_-) \rightarrow LCH_{\varepsilon_-}^{n-i+3}(\Lambda_-) \rightarrow .$$

*Here  $LCH_{\varepsilon_\pm}^i(\Lambda_\pm)$  is the linearized Legendrian contact cohomology of  $\Lambda_\pm$  over  $\mathbb{Z}_2$ , linearized with respect to the augmentation  $\varepsilon_\pm$ .  $\varepsilon_-$  is the augmentation induced by  $L_{\Lambda_-}$ , and  $\varepsilon_+$  is the augmentation induced by  $L$  and  $\varepsilon_-$ .*

We thank Joshua Sabloff and Lisa Traynor for pointing out the way to get the second long exact sequence in Theorem 1.4.

In [3], Chantraine described the way to construct Lagrangian cobordisms from Legendrian isotopies of Legendrian knots. We show that the construction of Chantraine works in high dimensions. More precisely, we prove the following:

**Theorem 1.5.** *Let  $\Lambda_-, \Lambda_+$  be two closed, orientable Legendrian submanifolds of  $\mathbb{R}^{2n+1}$  that are Legendrian isotopic, then there exists  $L$  such that  $\Lambda_- \prec_L^{ex} \Lambda_+$ .*

Front spinning is a procedure to construct a closed, orientable Legendrian submanifold  $\Sigma\Lambda \subset \mathbb{R}^{2n+3}$  from a closed, orientable Legendrian submanifold  $\Lambda \subset \mathbb{R}^{2n+1}$ . It was invented by Ekholm, Etnyre and Sullivan in [7]. The detailed description of this procedure will be provided in Section 5. We prove the following property of it:

**Theorem 1.6.** *Let  $\Lambda_-, \Lambda_+$  be two closed, orientable Legendrian submanifolds of  $\mathbb{R}^{2n+1}$ . If  $\Lambda_- \prec_L^{lag} \Lambda_+$ , then there exists  $\Sigma L$  such that  $\Sigma\Lambda_- \prec_{\Sigma L}^{lag} \Sigma\Lambda_+$ .*

Finally, we prove the following theorem:

**Theorem 1.7.** *There are infinitely many pairs of Lagrangian cobordant and not Legendrian isotopic Legendrian  $n$ -tori in  $\mathbb{R}^{2n+1}$ .*

## 2. PROOF OF THEOREM 1.2

Let  $n$  be even. First we recall the following proposition from [11]:

**Proposition 2.1** ([11]). *Let  $\Lambda$  be a closed, orientable, connected, chord generic Legendrian submanifold of  $\mathbb{R}^{2n+1}$ , where  $n$  is even. Then*

$$tb(\Lambda) = (-1)^{\frac{n}{2}+1} \frac{1}{2} \chi(\Lambda).$$

We now note that

$$(2.1) \quad \chi(\partial L) = 2\chi(L).$$

Equation 2.1 holds because the Euler characteristic of an even-dimensional boundary is twice the Euler characteristic of its bounded manifold, see Chapter 21 in [15]. We now observe that  $\partial L = \Lambda_+ \sqcup \Lambda_-$  and hence from Equation 2.1 we get that

$$(2.2) \quad \chi(\partial L) = \chi(\Lambda_+) + \chi(\Lambda_-) = 2\chi(L).$$

Then we use Proposition 2.1 and rewrite Equation 2.2 as

$$(2.3) \quad \chi(\Lambda_+) + \chi(\Lambda_-) = 2(-1)^{-\frac{n}{2}-1} (tb(\Lambda_+) + tb(\Lambda_-)) = 2\chi(L).$$

From Equation 2.3 it follows that

$$(2.4) \quad tb(\Lambda_+) + tb(\Lambda_-) = (-1)^{\frac{n}{2}+1} \chi(L).$$

This finishes the proof of Theorem 1.2 in the case when  $n$  is even.

We now prove (2). First we provide another definition of the Thurston-Bennequin number. Let  $\Lambda$  be a closed, orientable, connected, chord generic Legendrian submanifold of  $\mathbb{R}^{2n+1}$  and let  $c$  be a Reeb chord of  $\Lambda$  with end points  $a$  and  $b$  such that  $z(a) > z(b)$ . We define  $V_a := d\Pi(T_a\Lambda)$  and  $V_b := d\Pi(T_b\Lambda)$ . Given an orientation on  $\Lambda$ ,  $V_a$  and  $V_b$  are oriented  $n$ -dimensional transverse subspaces of  $\mathbb{R}^{2n}$ . If the orientation of  $V_a \oplus V_b$  agrees with that of  $\mathbb{R}^{2n}$ , then we say that the sign of  $c$ , we denote it by  $sign(c)$ , is  $+1$ , otherwise we say that it is  $-1$ . Then

$$(2.5) \quad tb(\Lambda) = \sum_c sign(c),$$

where the sum is taken over all Reeb chords  $c$  of  $\Lambda$ .

Using Formula 2.5, the following proposition was proven in [7]:

**Proposition 2.2** ([7]). *If  $\Lambda \subset \mathbb{R}^{2n+1}$  is a closed, orientable, connected, chord generic Legendrian submanifold, then*

$$tb(\Lambda) = (-1)^{\frac{(n-2)(n-1)}{2}} \sum_{c \in \mathcal{C}} (-1)^{|c|}.$$

We now remind the definition of linearized Legendrian contact homology complex of a closed, orientable, chord generic Legendrian submanifold  $\Lambda \subset \mathbb{R}^{2n+1}$ .

Let  $\mathcal{C}$  be the set of Reeb chords of  $\Lambda$ . Since  $\Lambda$  is generic,  $\mathcal{C}$  is a finite set. Let  $A_\Lambda$  be the vector space over  $\mathbb{Z}_2$  generated by the elements of  $\mathcal{C}$  and let  $\mathcal{A}_\Lambda$  be the unital tensor algebra over  $A_\Lambda$ , i.e.,

$$\mathcal{A}_\Lambda = \bigotimes_{k=0}^{\infty} A_\Lambda^{\otimes k}.$$

$\mathcal{A}_\Lambda$  is a differential graded algebra whose grading is denoted by  $|\cdot|$  and differential is denoted by  $\partial_\Lambda$ .  $\mathcal{A}_\Lambda$  is called a Legendrian contact homology differential graded algebra of  $\Lambda$ . For the definitions of  $|\cdot|$  and  $\partial_\Lambda$  we refer to Section 2 in [7].

Note that it is difficult to use Legendrian contact homology in practical applications, as it is the homology of an infinite dimensional noncommutative algebra with a nonlinear differential. One of the ways to extract useful information from the Legendrian contact homology differential graded algebra is to follow the Chekanov's method of linearization, which uses an augmentation  $\varepsilon : \mathcal{A}_\Lambda \rightarrow \mathbb{Z}_2$  to produce a finite-dimensional chain complex  $LC^\varepsilon(\Lambda)$  whose homology is denoted by  $LCH^\varepsilon(\Lambda)$  [4]. More precisely,  $\varepsilon$  is a graded algebra map  $\varepsilon : \mathcal{A}_\Lambda \rightarrow \mathbb{Z}_2$  that satisfy the following two conditions:

- (1)  $\varepsilon(1) = 1$ ;
- (2)  $\varepsilon \circ \partial_\Lambda = 0$ .

Consider the graded isomorphism  $\varphi^\varepsilon : \mathcal{A}_\Lambda \rightarrow \mathcal{A}_\Lambda$  defined by  $\varphi^\varepsilon(c) = c + \varepsilon(c)$ . This map defines a new differential  $\partial^\varepsilon(c) := \varphi^\varepsilon \circ \partial_\Lambda \circ (\varphi^\varepsilon)^{-1}(c)$  and  $LC^\varepsilon(\Lambda) := (A_\Lambda, \partial_1^\varepsilon)$ , where  $\partial_1^\varepsilon : A_\Lambda \rightarrow A_\Lambda$  is a 1-component of  $\partial^\varepsilon$ . We let  $LCH_\varepsilon(\Lambda)$  be the homology of the dual complex  $Hom(LC^\varepsilon(\Lambda), \mathbb{Z}_2)$ .

We now observe that for a closed, orientable, connected, chord generic Legendrian submanifold  $\Lambda$  we have

$$\chi(LC^\varepsilon(\Lambda)) = \sum_{i \in \mathbb{Z}} (-1)^i rk(LCH_i^\varepsilon(\Lambda)) = \sum_{i \in \mathbb{Z}} (-1)^i rk(LC_i^\varepsilon(\Lambda)).$$

Here  $\varepsilon$  is any augmentation  $\varepsilon : \mathcal{A}_\Lambda \rightarrow \mathbb{Z}_2$ . Hence, we get that

$$(2.6) \quad \chi(LC^\varepsilon(\Lambda)) = \sum_{c \in \mathcal{C}} (-1)^{|c|}.$$

Therefore, from Proposition 2.2 and Equation 2.6 we get

$$(2.7) \quad tb(\Lambda) = (-1)^{\frac{(n-2)(n-1)}{2}} \chi(LC^\varepsilon(\Lambda)).$$

We now observe that

$$(2.8) \quad \chi(LC_\varepsilon(\Lambda)) = \chi(LC^\varepsilon(\Lambda)).$$

Hence, from Equations 2.7 and 2.8 it follows that

$$(2.9) \quad tb(\Lambda) = (-1)^{\frac{(n-2)(n-1)}{2}} \chi(LC_\varepsilon(\Lambda)).$$

We now remind the following fact described by Ekholm in [6], which comes from certain observations of Seidel in wrapped Floer homology [1], [13].

**Fact 2.3** ([6]). *Let  $\Lambda$  be a closed, orientable, connected, chord generic Legendrian submanifold of  $\mathbb{R}^{2n+1}$  and  $\emptyset \prec_{L_\Lambda}^{ex} \Lambda$ . Then*

$$H_{n-i+2}(L_\Lambda) \simeq LCH_\varepsilon^i(\Lambda).$$

Here  $\varepsilon$  is the augmentation induced by  $L_\Lambda$ .

Observe that Ekholm in [6] provided a fairly complete sketch of proof of Fact 2.3.

Since  $\Lambda_-$  is connected, and  $L, L_{\Lambda_-}$  are exact Lagrangian cobordisms in the symplectization of  $\mathbb{R}^{2n+1}$  such that  $(-\infty)$ -boundary of  $L$ , which is  $\Lambda_-$ , agrees with  $(+\infty)$ -boundary of  $L_{\Lambda_-}$ , then  $L$  and  $L_{\Lambda_-}$  can be joined to the exact Lagrangian cobordism  $L_{\Lambda_+}$  in the symplectization of  $\mathbb{R}^{2n+1}$ , where  $L_{\Lambda_+}$  is obtained by gluing the positive end of  $L_{\Lambda_-}$  to the negative end of  $L$ . Since the  $-\infty$ -boundary of  $L_{\Lambda_-}$  is empty, the  $-\infty$ -boundary of  $L_{\Lambda_+}$  is also empty.

From Fact 2.3 it follows that

$$(2.10) \quad LCH_{\varepsilon_\pm}^i(\Lambda_\pm) \simeq H_{n-i+2}(L_{\Lambda_\pm}).$$

Here  $\varepsilon_-$  is the augmentation induced by  $L_{\Lambda_-}$  and  $\varepsilon_+$  is the augmentation induced by  $L$  and  $\varepsilon_-$ . Observe that since  $n$  is odd,  $n - i + 2$  and  $i$  have different parity. Hence, using Formula 2.10 we get that

$$(2.11) \quad \chi(LC_{\varepsilon_\pm}(\Lambda_\pm)) = -\chi(L_{\Lambda_\pm}).$$

We now observe that Equations 2.9 and 2.11 imply that

$$(2.12) \quad tb(\Lambda_\pm) = (-1)^{\frac{(n-2)(n-1)}{2}+1} \chi(L_{\Lambda_\pm}).$$

Here we see that

$$(2.13) \quad \chi(L_{\Lambda_+}) = \chi(L) + \chi(L_{\Lambda_-}) - \chi(\Lambda_-).$$

Since  $\Lambda_-$  is a closed odd-dimensional manifold,  $\chi(\Lambda_-) = 0$ . Thus, we can rewrite Equation 2.13 as

$$(2.14) \quad \chi(L_{\Lambda_+}) - \chi(L_{\Lambda_-}) = \chi(L).$$

We finally use Equations 2.12 and 2.14 and get

$$(2.15) \quad tb(\Lambda_+) - tb(\Lambda_-) = (-1)^{\frac{(n-2)(n-1)}{2}+1} \chi(L).$$

This finishes the proof of Theorem 1.2 when  $n$  is odd.

*Remark 2.4.* Note that when  $n = 1$  Equation 2.15 can be written as

$$tb(\Lambda_+) - tb(\Lambda_-) = -\chi(L),$$

which coincides with the formula from Theorem 1.2 in [3].

## 3. PROOF OF REMARK 1.3

We first consider the case when  $n$  is even. Now, as in the proof of Theorem 1.2, we use Proposition 2.1. It implies that

$$tb(\Lambda) = (-1)^{\frac{n}{2}+1} \frac{1}{2} \chi(L_\Lambda).$$

We then use the fact that  $\chi(\Lambda) = \chi(\partial L) = 2\chi(L)$  for even  $n$ , see Chapter 21 in [15], which implies that

$$tb(\Lambda) = (-1)^{\frac{n}{2}+1} \chi(L_\Lambda).$$

We now consider the case when  $n$  is odd. As in the proof of Theorem 1.2, we use Proposition 2.2, which says that

$$tb(\Lambda) = (-1)^{\frac{(n-2)(n-1)}{2}} \sum_{c \in \mathcal{C}} (-1)^{|c|}.$$

Observe that for a closed, orientable, chord generic Legendrian submanifold  $\Lambda$

$$\begin{aligned} (3.1) \quad tb(\Lambda) &= (-1)^{\frac{(n-2)(n-1)}{2}} \chi(LC^\varepsilon(\Lambda)) = \sum_{i \in \mathbb{Z}} (-1)^i rk(LCH_i^\varepsilon(\Lambda)) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i rk(LC_i^\varepsilon(\Lambda)). \end{aligned}$$

Here  $\varepsilon$  is any augmentation  $\varepsilon : \mathcal{A}_\Lambda \rightarrow \mathbb{Z}_2$ . We now assume that  $\varepsilon$  is induced from  $L_\Lambda$ . Hence, we see that from Fact 2.3 and Formula 3.1 it follows that

$$tb(\Lambda) = (-1)^{\frac{(n-2)(n-1)}{2}+1} \chi(L_\Lambda).$$

This finishes the proof of Remark 1.3.

## 4. PROOF OF THEOREM 1.4

First we construct an exact Lagrangian filling of  $\Lambda_+$ . We do it the same way as in the proof of Theorem 1.2, namely  $L_{\Lambda_+}$  is obtained by gluing the positive end of  $L_{\Lambda_-}$  to the negative end of  $L$  in the symplectization of  $\mathbb{R}^{2n+1}$ .

We now use the Mayer-Vietoris long exact sequence for  $L_{\Lambda_-}, L \subset L_{\Lambda_+}$ . We possibly extend  $L_{\Lambda_-}$  and  $L$  in such a way that  $L_{\Lambda_-} \cap L = \mathbb{R} \times \Lambda_-$ . Hence, the Mayer-Vietoris long exact sequence can be written as

$$\rightarrow H_i(\mathbb{R} \times \Lambda_-) \rightarrow H_i(L) \oplus H_i(L_{\Lambda_-}) \rightarrow H_i(L_{\Lambda_+}) \rightarrow H_{i-1}(\mathbb{R} \times \Lambda_-) \rightarrow .$$

Now we note that  $H_i(\mathbb{R} \times \Lambda_-) \simeq H_i(\Lambda_-)$  for all  $i$ . Hence, we can rewrite the Mayer-Vietoris long exact sequence as

$$(4.1) \quad \rightarrow H_i(\Lambda_-) \rightarrow H_i(L) \oplus H_i(L_{\Lambda_-}) \rightarrow H_i(L_{\Lambda_+}) \rightarrow H_{i-1}(\Lambda_-) \rightarrow .$$

We recall that Fact 2.3 says that

$$(4.2) \quad LCH_{\varepsilon_\pm}^i(\Lambda_\pm) \simeq H_{n-i+2}(L_{\Lambda_\pm}).$$

Here  $\varepsilon_-$  is the augmentation induced by  $L_{\Lambda_-}$  and  $\varepsilon_+$  is the augmentation induced by  $L$  and  $\varepsilon_-$ . We change the indices in Formula 4.2 and write it as

$$(4.3) \quad H_i(L_{\Lambda_{\pm}}) \simeq LCH_{\varepsilon_{\pm}}^{n-i+2}(\Lambda_{\pm}).$$

Using Formula 4.3, we rewrite Mayer-Vietoris long exact sequence 4.1 as

$$(4.4) \quad \rightarrow H_i(\Lambda_-) \rightarrow H_i(L) \oplus LCH_{\varepsilon_-}^{n-i+2}(\Lambda_-) \rightarrow LCH_{\varepsilon_+}^{n-i+2}(\Lambda_+) \rightarrow H_{i-1}(\Lambda_-) \rightarrow .$$

We now write the long exact sequence for the pair  $(L_{\Lambda_-}, L_{\Lambda_+})$

$$(4.5) \quad \rightarrow H_i(L_{\Lambda_-}) \rightarrow H_i(L_{\Lambda_+}) \rightarrow H_i(L_{\Lambda_+}, L_{\Lambda_-}) \rightarrow H_{i-1}(L_{\Lambda_-}) \rightarrow .$$

Using Formula 4.3, we rewrite long exact sequence 4.5 as

$$(4.6) \quad \rightarrow LCH_{\varepsilon_-}^{n-i+2}(\Lambda_-) \rightarrow LCH_{\varepsilon_+}^{n-i+2}(\Lambda_+) \rightarrow H_i(L_{\Lambda_+}, L_{\Lambda_-}) \rightarrow LCH_{\varepsilon_-}^{n-i+3}(\Lambda_-) \rightarrow .$$

Finally, we use the excision theorem for  $L_{\Lambda_+}, L \subset L_{\Lambda_+}$  and see that

$$H_i(L_{\Lambda_+}, L_{\Lambda_-}) \simeq H_i(L, \Lambda_-).$$

Hence, we can write long exact sequence 4.6 as

$$(4.7) \quad \rightarrow LCH_{\varepsilon_-}^{n-i+2}(\Lambda_-) \rightarrow LCH_{\varepsilon_+}^{n-i+2}(\Lambda_+) \rightarrow H_i(L, \Lambda_-) \rightarrow LCH_{\varepsilon_-}^{n-i+3}(\Lambda_-) \rightarrow .$$

This finishes the proof of Theorem 1.4.

*Remark 4.1.* Note that under the conditions of Theorem 1.4, if  $H_i(\Lambda_-) = H_{i-1}(\Lambda_-) = 0$  for some  $i$ , say when  $\Lambda_- = S^n$  and  $i, i-1 \neq 0, n$ , then long exact sequence 4.4 implies that

$$LCH_{\varepsilon_+}^{n-i+2}(\Lambda_+) \simeq H_i(L) \oplus LCH_{\varepsilon_-}^{n-i+2}(\Lambda_-).$$

Hence, for such  $i$  we get that

$$H_i(L) \simeq LCH_{\varepsilon_+}^{n-i+2}(\Lambda_+)/LCH_{\varepsilon_-}^{n-i+2}(\Lambda_-).$$

*Remark 4.2.* Note that we can rewrite long exact sequences 4.4 and 4.7 using the relative symplectic field theory of  $((\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha)), L_{\Lambda_{\pm}})$ , since

$$(4.8) \quad E_1^i((\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha)), L_{\Lambda_{\pm}}) \simeq LCH_{\varepsilon_{\pm}}^i(\Lambda_{\pm})$$

over  $\mathbb{Z}_2$ . For the definition of the relative symplectic field theory we refer to [5], for the details about the isomorphism described in Formula 4.8 we refer to [6] (we observe that since  $L_{\Lambda_{\pm}}$  are connected, the associated spectral sequences have only one level).

## 5. EXAMPLES

In this section, we describe a few examples of Lagrangian cobordisms. These examples are based on the works of Chantraine [3], Ekholm, Etnyre and Sullivan [7], and Ekholm, Honda and K  lm  n [9]. For the constructions of Lagrangian cobordisms based on the generating families technique we refer to Sabloff and Traynor [18].



*Example 5.1 (Legendrian isotopy).* Let  $\Lambda_-, \Lambda_+ \subset \mathbb{R}^{2n+1}$  be two closed, orientable Legendrian submanifolds which are Legendrian isotopic, i.e., there is a smooth isotopy of a closed manifold  $\Lambda$  to  $\mathbb{R}^{2n+1}$  given by  $\varphi : \Lambda \times [0, 1] \rightarrow \mathbb{R}^{2n+1}$  such that  $\Lambda_\nu := \varphi(\Lambda, \nu)$  is Legendrian for all  $\nu \in [0, 1]$ ,  $\Lambda_- = \Lambda_0$  and  $\Lambda_+ = \Lambda_1$ . We now construct  $L$  such that  $\Lambda_- \prec_L^{ex} \Lambda_+$ . Observe that in the construction below one can omit the assumption that  $\Lambda_-, \Lambda_+, L$  are connected. In the case of Legendrian knots in  $\mathbb{R}^3$ , the construction of  $L$  was described by Chantraine, see Theorem 1.1 in [3]. In our case, the construction of Chantraine can be described in the following way:

- (1) We note that  $\mathbb{R} \times \Lambda_-$  is an exact Lagrangian submanifold of  $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha))$ .
- (2) Theorem 2.6.2 from [14] implies that there is a compactly supported one-parameter family of contactomorphisms  $f_\nu$  which realizes the isotopy  $(\Lambda_\nu)_{\nu \in [0, 1]}$ .
- (3) Proposition 2.2 from [3] implies that a contactomorphism of  $\mathbb{R}^{2n+1}$  lifts to a Hamiltonian diffeomorphism of the symplectization  $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha))$ .
- (4) Let  $H$  be a Hamiltonian on  $\mathbb{R} \times \mathbb{R}^{2n+1}$  whose flow realizes the lifts of  $f_\nu$ 's. The existence of  $H$  follows from (3). Following Chantraine, we construct

$$H' : \mathbb{R} \times \mathbb{R}^{2n+1} \times [0, 1] \rightarrow \mathbb{R}$$

such that

$$H'(t, x, \nu) = \begin{cases} H(t, x, \nu), & \text{for } t > T; \\ 0, & \text{for } t < -T. \end{cases}$$

Here  $T \gg 0$ .

- (5) Let  $\phi^\nu$  be the Hamiltonian flow of  $H'$ . We now observe that  $\phi^1(\mathbb{R} \times \Lambda_-)$  coincides with  $\mathbb{R} \times \Lambda_-$  near  $-\infty$  and with  $\mathbb{R} \times \Lambda_+$  near  $\infty$ .
- (6) Since  $\mathbb{R} \times \Lambda_-$  is exact and  $\phi^1$  is a Hamiltonian diffeomorphism,  $L := \phi^1(\mathbb{R} \times \Lambda_-)$  is exact.

This finishes the proof of Theorem 1.5.

Before we discuss the next example, we briefly recall a few facts about Lagrangian cobordisms between Legendrian knots in  $\mathbb{R}^3$ .

*Theorem 5.2 ([9, 10]).* *There exists an exact Lagrangian cobordism for the following:*

- (1) *Legendrian isotopy,*
- (2) *0-resolution at a contractible crossing in the Lagrangian projection,*
- (3) *capping off a  $tb = -1$  unknot with a disk.*

See Figure 1 for the 0-resolution on the Lagrangian projection.

Following Ekholm, Honda and Kálmán, we say that *contractible crossing* of  $\Lambda$  is a crossing so that  $z_1 - z_0$  can be shrunk to zero without affecting the other crossings. (Here  $z_1$  is the  $z$ -coordinate on the upper strand and  $z_0$  is the  $z$ -coordinate on the lower strand.)

*Conjecture 5.3 ([9, 10]).* *If  $\emptyset \prec_{L_\Lambda}^{ex} \Lambda$ , then  $L_\Lambda$  is obtained by stacking exact Lagrangians cobordisms described in Theorem 5.2.*

*Example 5.4 (Front spinning).* The following construction is based on the front spinning method invented by Ekholm, Etnyre and Sullivan in [7].

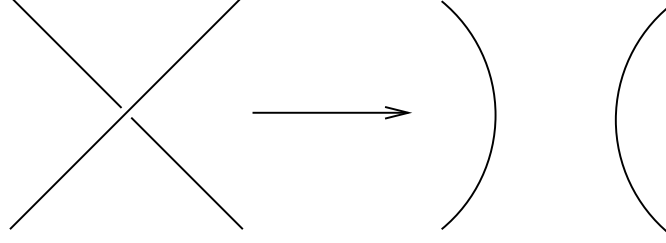


FIGURE 1. The 0-resolution on the Lagrangian projection.

First we recall the notion of front projection. *Front projection* is a map  $\Pi_F$  from  $\mathbb{R}^{2n+1}$  to  $\mathbb{R}^{n+1}$  defined by

$$\Pi_F(x_1, y_1, \dots, x_n, y_n, z) = (x_1, x_2, \dots, x_n, z).$$

Let  $\Lambda$  be a closed, orientable Legendrian submanifold of  $\mathbb{R}^{2n+1}$  parametrized by  $f_\Lambda : \Lambda \rightarrow \mathbb{R}^{2n+1}$  and we write

$$f_\Lambda(p) = (x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p))$$

for  $p \in \Lambda$ . The front projection of  $\Lambda$  is parametrized by  $\Pi_F \circ f_\Lambda$  and we have

$$\Pi_F \circ f_\Lambda(p) = (x_1(p), x_2(p), \dots, x_n(p), z(p)).$$

Without loss of generality we can assume that  $x_1(p) > 0$  for all  $p \in \Lambda$ . We now embed  $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+2}$  via

$$(x_1, \dots, x_n, z) \rightarrow (x_0 = 0, x_1, \dots, x_n, z)$$

and construct the suspension of  $\Lambda$ , we denote it by  $\Sigma\Lambda$ , such that  $\Pi_F(\Sigma\Lambda)$  is obtained from  $\Pi_F(\Lambda)$  by rotating it around the subspace  $x_0 = x_1 = 0$ .  $\Pi_F(\Sigma\Lambda)$  can be parametrized by  $(x_1(p) \sin \theta, x_1(p) \cos \theta, x_2(p), \dots, x_n(p), z(p))$  with  $\theta \in S^1$  and is the front projection of a Legendrian embedding  $\Lambda \times S^1 \rightarrow \mathbb{R}^{2n+3}$ . For the properties of  $\Sigma\Lambda$  we refer to Lemma 4.16 in [7].

Let  $\Lambda_-, \Lambda_+$  be two closed, orientable Legendrian submanifolds of  $\mathbb{R}^{2n+1}$  such that

$$(5.1) \quad \Lambda_\pm \subset \{(x_1, y_1, \dots, x_n, y_n, z) \in \mathbb{R}^{2n+1} \mid x_1 > 0\}$$

and  $\Lambda_- \prec_L^{lag} \Lambda_+$ . Let  $L$  be parametrized by  $f_L : L \rightarrow \mathbb{R}^{2n+2}$

$$f_L(p) = (t(p), x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p)).$$

Without loss of generality we assume that  $x_1(p) > 0$  for all  $p$  (Property 5.1 implies that  $\{f_L(p) \mid x_1(p) \leq 0\}$  is compact and we can translate  $L$  so that  $x_1(p) > 0$  for all  $p$ ). Then we construct a Lagrangian cobordism from  $\Sigma\Lambda_-$  to  $\Sigma\Lambda_+$  that we call  $\Sigma L$ . We define  $\Sigma L$  to be parametrized by  $f_{\Sigma L} : L \times S^1 \rightarrow \mathbb{R} \times \mathbb{R}^{2n+3}$  with

$$f_{\Sigma L}(p, \theta) = (t(p), x_1(p) \sin \theta, y_1(p) \sin \theta, x_1(p) \cos \theta, y_1(p) \cos \theta, x_2(p), \dots, z(p)).$$

Here  $p \in L$  and  $\theta \in S^1$ .

We now show that  $\Sigma L$  is really a Lagrangian cobordism from  $\Sigma\Lambda_-$  to  $\Sigma\Lambda_+$ . Let

$$\begin{aligned}\Lambda_+^{T_L} &:= \{(x_0, \dots, y_n, z) \mid (T_L, x_0, \dots, y_n, z) \in f_{\Sigma L}(\Sigma L) \cap (\{T_L\} \times \mathbb{R}^{2n+3})\} \text{ and} \\ \Lambda_-^{T_L} &:= \{(x_0, \dots, y_n, z) \mid (-T_L, x_0, \dots, y_n, z) \in f_{\Sigma L}(\Sigma L) \cap (\{-T_L\} \times \mathbb{R}^{2n+3})\}.\end{aligned}$$

We now note that from the definition of  $T_L$  it follows that

$$\begin{aligned}f_{\Sigma L}(\Sigma L) \cap ([T_L, \infty) \times \mathbb{R}^{2n+3}) &= [T_L, \infty) \times \Lambda_+^{T_L} \text{ and} \\ f_{\Sigma L}(\Sigma L) \cap ((-\infty, -T_L] \times \mathbb{R}^{2n+3}) &= (-\infty, -T_L] \times \Lambda_-^{T_L}.\end{aligned}$$

In addition, we observe that  $\Lambda_{\pm}^{T_L} \subset \mathbb{R}^{2n+3}$  can be parametrized by

$$f_{\Lambda_{\pm}^{T_L}} : \Lambda_{\pm} \times S^1 \rightarrow \mathbb{R}^{2n+3}$$

such that

$$f_{\Lambda_{\pm}^{T_L}}(p, \theta) = (x_1(p) \sin \theta, y_1(p) \sin \theta, x_1(p) \cos \theta, y_1(p) \cos \theta, x_2(p), \dots, z(p)).$$

Here  $p \in \Lambda_{\pm} \subset \partial L$  and  $\theta \in S^1$ . We now prove that  $\Lambda_{\pm}^{T_L}$  coincides with  $\Sigma\Lambda_{\pm}$ . It is clear that  $\Pi_F(\Lambda_{\pm}^{T_L}) = \Pi_F(\Sigma\Lambda_{\pm})$ . It remains to prove that  $\Lambda_{\pm}^{T_L}$  is a Legendrian submanifold of  $\mathbb{R}^{2n+3}$ .

It is easy to see that

$$\begin{aligned}(5.2) \quad f_{\Lambda_{\pm}^{T_L}}^*(dz - \sum_{i=0}^n y_i dx_i) &= dz(p) - \sum_{i=2}^n y_i(p) dx_i(p) \\ &\quad - y_1(p)(\sin^2 \theta + \cos^2 \theta) dx_1(p) + (y_1(p)x_1(p) \sin \theta \cos \theta - y_1(p)x_1(p) \sin \theta \cos \theta) d\theta.\end{aligned}$$

Since  $\Lambda_{\pm}$  is Legendrian submanifold of  $\mathbb{R}^{2n+1}$  and hence  $f_{\Lambda_{\pm}}^*(dz - \sum_{i=1}^n y_i dx_i) = 0$ , we have that

$$(5.3) \quad y_1(p) dx_1(p) = dz(p) - \sum_{i=2}^n y_i(p) dx_i(p).$$

Hence, Formulas 5.2 and 5.3 imply that

$$(5.4) \quad f_{\Lambda_{\pm}^{T_L}}^*(dz - \sum_{i=0}^n y_i dx_i) = 0.$$

Since

$$f_{\Lambda_{\pm}}(p) := (x_1(p), \dots, y_n(p), z(p))$$

with  $p \in \Lambda_{\pm} \subset \partial L$  is a parametrization of an embedded submanifold of dimension  $n$ , and  $x_1(p) > 0$  for  $p \in \Lambda_{\pm} \subset \partial L$ , one easily sees that

$$f_{\Lambda_{\pm}^{T_L}}(p) = (x_1(p) \sin \theta, y_1(p) \sin \theta, x_1(p) \cos \theta, y_1(p) \cos \theta, x_2(p), \dots, z(p))$$

with  $p \in \Lambda_{\pm}, \theta \in S^1$  is a parametrization of an embedded submanifold of dimension  $n+1$ . Thus, using Formula 5.4 we see that  $\Lambda_{\pm}^{T_L}$  is an embedded Legendrian submanifold of  $\mathbb{R}^{2n+3}$  whose front projection coincides with  $\Pi_F(\Sigma\Lambda_{\pm})$ . Thus, we get that  $\Lambda_{\pm}^{T_L} = \Sigma\Lambda_{\pm}$ .

We now note that

$$\begin{aligned}
 (5.5) \quad f_{\Sigma L}^*(e^t(dz - \sum_{i=0}^n y_i dx_i)) &= e^t(dt(p) \wedge dz(p) - \sum_{i=2}^n dy_i(p) \wedge dx_i(p) \\
 &\quad - \sum_{i=2}^n y_i(p) dt(p) \wedge dx_i(p) - (y_1(p)(\sin^2 \theta + \cos^2 \theta) dt(p) \wedge dx_1(p) \\
 &\quad + (\sin^2 \theta + \cos^2 \theta) dy_1(p) \wedge dx_1(p) + (\sin^2 \theta + \cos^2 \theta) x_1(p) y_1(p) d\theta \wedge d\theta \\
 &\quad + (y_1(p) x_1(p) \sin \theta \cos \theta - y_1(p) x_1(p) \sin \theta \cos \theta) dt(p) \wedge d\theta \\
 &\quad + (y_1(p) \sin \theta \cos \theta - y_1(p) \sin \theta \cos \theta) d\theta \wedge dx_1(p) \\
 &\quad + (x_1(p) \sin \theta \cos \theta - x_1(p) \sin \theta \cos \theta) dy_1(p) \wedge d\theta)).
 \end{aligned}$$

In addition, observe that

$$\begin{aligned}
 (5.6) \quad e^t(dt(p) \wedge dz(p) - \sum_{i=2}^n dy_i(p) \wedge dx_i(p) - \sum_{i=2}^n y_i(p) dt(p) \wedge dx_i(p)) \\
 = e^t(y_1(p) dt(p) \wedge dx_1(p) + dy_1(p) \wedge dx_1(p)).
 \end{aligned}$$

Hence, Formulas 5.5 and 5.6 imply that

$$(5.7) \quad f_{\Sigma L}^*(e^t(dz - \sum_{i=0}^n y_i dx_i)) = 0.$$

Since

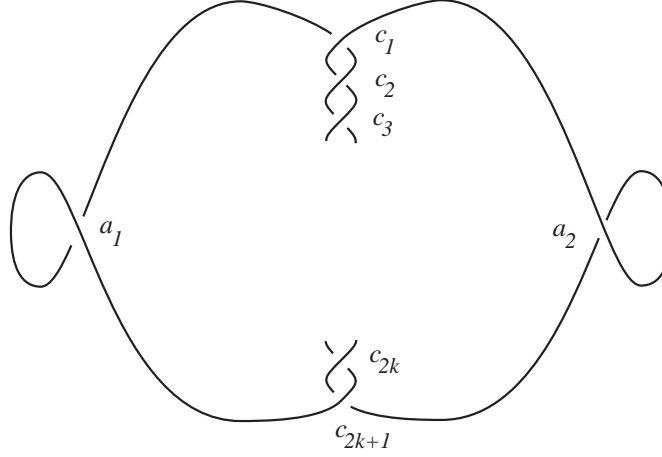
$$f_L(p) = (t(p), x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p)),$$

where  $p \in L$ , is a parametrization of an embedded cobordism of dimension  $n + 1$  and  $x_1(p) > 0$  for  $p \in L$ , one easily sees that

$$f_{\Sigma L}(p, \theta) = (t(p), x_1(p) \sin \theta, y_1(p) \sin \theta, x_1(p) \cos \theta, y_1(p) \cos \theta, x_2(p), \dots, z(p)),$$

where  $p \in L$  and  $\theta \in S^1$ , is a parametrization of an embedded cobordism of dimension  $n + 2$ . Hence, we use Formula 5.7 and see that  $\Sigma L$  is really an embedded Lagrangian cobordism from  $\Sigma \Lambda_-$  to  $\Sigma \Lambda_+$ . This finishes the proof of Theorem 1.6.

We now use Example 2 to get infinitely many pairs of Lagrangian cobordant and not Legendrian isotopic Legendrian  $n$ -tori in  $\mathbb{R}^{2n+1}$ . We first recall that Theorem 5.2 says that 0-resolution at a contractible crossing in the Lagrangian projection can be realized as a Lagrangian cobordism. Let  $T_{2k+1}$  be the Legendrian torus knot from Example 4.18 in [7], see Figure 2 for the Lagrangian projection of  $T_{2k+1}$ . One observes that all the crossings in the middle part of the Lagrangian projection are contractible, see [10] for the case of  $T_3$ , and hence one can get  $T_{2k-1}$  from  $T_{2k+1}$  by contracting  $c_{2k+1}$  and then  $c_{2k}$ . Let  $L_{2k}^{2k+1}$  be a Lagrangian cobordism which corresponds to the 0-resolution at  $c_{2k+1}$  and let  $L_{2k-1}^{2k}$  be a Lagrangian cobordism from  $T_{2k-1}$  to  $T_{2k}$  which corresponds to the resolution of  $c_{2k}$ . Then we stack  $L_{2k}^{2k+1}$  and  $L_{2k-1}^{2k}$  and get a Lagrangian cobordism that we call  $L_{2k-1}^{2k+1}$  such that  $T_{2k-1} \prec_{L_{2k-1}^{2k+1}} T_{2k+1}$ . If we stack  $L_{2i-1}^{2i+1}$ 's we get a Lagrangian cobordism  $L_{2j+1}^{2k+1}$

FIGURE 2. The knot  $T_{2k+1}$ , cf Figure 13 in [7].

such that  $T_{2j+1} \prec_{L_{2j+1}^{2k+1}} T_{2k+1}$  for  $k > j$ . We use the construction described in Example 2 and get  $\Sigma^n T_{2j+1} \prec_{\Sigma^n L_{2j+1}^{2k+1}} \Sigma^n T_{2k+1}$  for  $k > j$ . We now recall that Ekholm, Etnyre and Sullivan proved that  $\Sigma^n T_{2j+1}$  is not Legendrian isotopic to  $\Sigma^n T_{2k+1}$  for  $k > j + 1$  and  $j \in \mathbb{N}$ , see Theorem 4.19 in [7].

Hence, we get infinitely many pairs of Lagrangian cobordant and not Legendrian isotopic Legendrian  $n$ -tori in  $\mathbb{R}^{2n+1}$ . This finishes the proof of Theorem 1.7.

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